NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE

No. 1815

ON COMPRESSIBILITY CORRECTIONS FOR SUBSONIC FLOW OVER BODIES OF REVOLUTION

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SUMMARY

A study of the subsonic flow past an infinitely long corrugated circular cylinder is presented to show the relation between two-dimensional and axisymmetrical flow. In fact, a solution is obtained which contains as limiting cases both the Prandtl-Glauert correction for two-dimensional flow and the Göthert correction for flow past slender bodies of revolution. Included in the paper are velocity-correction formulas for a cylinder with a single bump and for a corrugated cylinder in the presence of walls.

INTRODUCTION

The present paper is concerned with the form of the compressibility corrections for subsonic flow which follow from the linear-perturbation theory. It is now well known that there are essential differences between the compressibility corrections for two-dimensional flow and the corresponding corrections for flow about slender bodies of revolution. This problem has been the subject of several papers by various authors. (See, for example, references 1 and 2.)

The analysis presented herein shows that the relation between two-dimensional and axisymmetrical flow can be clearly demonstrated in the solution for the flow past an infinitely long corrugated cylinder.[1] In fact, a solution is obtained which contains as limiting cases both the Prandtl-Glauert correction for two-dimensional flow and the Göthert correction for flow past slender bodies of revolution. Although the results for these two limiting cases are already known, the result obtained in the present paper shows the nature of the transition from one limiting case to the other. The nature of this transition has been treated from a different point of view in reference 4, where the bodies considered consisted of a family of ellipsoids ranging from the ellipsoid of revolution to the infinitely long elliptic cylinder. It is of interest that the present example is a natural extension of the two-dimensional wavy wall treated by Ackeret in a classical paper (reference 5).

[1]Dr. C. C. Lin has pointed out to the author that flow past a corrugated circular cylinder has previously been considered by Th. von Kármán (reference 3) in a different connection as an example of the calculation of wave drag for supersonic flow past bodies of revolution.

It should be mentioned that the results presented herein were obtained in June 1948, while the author was associated with the National Advisory Committee for Aeronautics at Langley Air Force Base, Va.

AXISYMMETRICAL LINEAR-PERTURBATION FLOW PAST A CORRUGATED CYLINDER

Let $u + U$ and v be components of fluid velocity in the axial and radial directions, respectively. Let $\phi(x,r)$ be the pertubation velocity potential in terms of which

$$\left.\begin{aligned} u &= \frac{\partial\phi}{\partial x} \\ v &= \frac{\partial\phi}{\partial r} \end{aligned}\right\} \tag{1}$$

where x and r are the axial and radial directions, respectively. The linearized differential equation for ϕ is

$$\left(1 - M_\infty{}^2\right) \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0 \tag{2}$$

where M_∞ is the undisturbed stream Mach number.

Let $r = a + f(x)$ be the equation of the meridian profile of the body of revolution such that $|f(x)| \ll a$ and $|f'(x)| \ll 1$. The boundary condition at the surface of the body of revolution is then of the following form:

$$\left.\begin{aligned} r &= a \\ \frac{\partial\phi}{\partial r} &= Uf'(x) \end{aligned}\right\} \tag{3}$$

Consider now the particular case in which

$$f(x) = l\delta \cos\left(\pi\frac{x}{l}\right) \tag{4}$$

where δ is the thickness ratio η/l of the ripple (η is the amplitude and $2l$ is the wave length of the ripple (fig. 1)). An appropriate solution of differential equation (2) is

$$\phi = \sum_{n=1}^{\infty} \sin\frac{n\pi}{l}x \left[A_n I_0\left(\sqrt{1-M_\infty^2}\,\frac{n\pi}{l}r\right) + B_n K_0\left(\sqrt{1-M_\infty^2}\,\frac{n\pi}{l}r\right)\right] \tag{5}$$

where I_0 and K_0 are modified Bessel functions of order zero and A_n and B_n are arbitrary constants. The following properties of I_0 and K_0 are needed:

$$\left.\begin{array}{l} I_0'(x) = I_1(x);\ K_0'(x) = -K_1(x) \\ x \ll 1 \begin{cases} I_0(x) \approx 1,\ K_0(x) \approx -\log_e x + 0.116 \\ I_1(x) \approx \dfrac{x}{2},\ K_1(x) \approx \dfrac{1}{x} \end{cases} \\ x \gg 1 \begin{cases} I_0(x) \approx I_1(x) \approx \dfrac{e^x}{\sqrt{2\pi x}} \\ K_0(x) \approx K_1(x) \approx \sqrt{\dfrac{\pi}{2x}}\, e^{-x} \end{cases} \end{array}\right\} \tag{6}$$

When the boundary of the cylinder is given by equation (4), only the first term of the series (equation (5)) is needed.

For the body of revolution in an unlimited air stream, the asymptotic behavior of the functions I_n requires that the coefficients A_n vanish. From equations (5), (4), and (3), it follows that the form of the perturbation potential caused by the ripple is

$$\phi = \frac{Ul\delta}{\sqrt{1-M_\infty^2}} \frac{K_0\left(\sqrt{1-M_\infty^2}\ \pi\frac{r}{l}\right)}{K_1\left(\sqrt{1-M_\infty^2}\ \pi\frac{a}{l}\right)} \sin\left(\pi\frac{x}{l}\right) \tag{7}$$

Equation (7) leads to the following expression for the axial velocity u at the surface of the body of revolution:

$$u(x,a) = \frac{\pi U\delta}{\sqrt{1-M_\infty^2}} \frac{K_0\left(\sqrt{1-M_\infty^2}\ \pi\frac{a}{l}\right)}{K_1\left(\sqrt{1-M_\infty^2}\ \pi\frac{a}{l}\right)} \cos\left(\pi\frac{x}{l}\right) \tag{8}$$

The ratio of $u(x,a)$ for compressible and for incompressible flow is then

$$\frac{u_c}{u_i} = \frac{1}{\sqrt{1-M_\infty^2}} \frac{K_0\left(\sqrt{1-M_\infty^2}\ \pi\frac{a}{l}\right)}{K_1\left(\sqrt{1-M_\infty^2}\ \pi\frac{a}{l}\right)} \frac{K_1\left(\pi\frac{a}{l}\right)}{K_0\left(\pi\frac{a}{l}\right)} \tag{9}$$

From equations (9) and (6) it follows, in particular, that, when $\pi\sqrt{1-M_\infty^2}\ \frac{a}{l} \gg 1$;

$$\frac{u_c}{u_i} \approx \frac{1}{\sqrt{1-M_\infty^2}} \tag{10a}$$

and, when $\pi\frac{a}{l} \ll 1$,

$$\frac{u_c}{u_i} \approx 1 + \frac{\log_e \sqrt{1-M_\infty^2}}{1.03 + \log_e\frac{a}{l}} \tag{10b}$$

Equation (10a) is of the form of the Prandtl-Glauert correction and equation (10b) is of the form of the Göthert correction. The transition between the two forms is supplied by equation (9). Figure 2 shows the relation between the results of equations (9) and (10) for a given value of the undisturbed stream Mach number ($M_\infty = 0.866$).

Note that the validity of the foregoing formulas is governed by the following two restrictions. First, the use of the linearized boundary condition (equation (3)) requires that $\delta = \frac{\eta}{l} \ll 1$ and, second, that $\delta \ll \frac{a}{l}$. When the second of these two restrictions is not satisfied, it is necessary to satisfy the boundary condition along the line $r = a + \eta \cos\left(\pi\frac{x}{l}\right)$ rather than along the line $r = a$ and hence the velocity correction formulas depend on two length ratios $\frac{a}{l}$ and $\frac{\eta}{l}$.

VELOCITY CORRECTION FORMULA FOR CYLINDER WITH BUMP

From the foregoing results the corresponding results may be deduced for a body of revolution having a meridian profile given by an equation of the form

$$r = a + \int_0^\infty \eta(\lambda)\ \cos(\lambda x)d\lambda \tag{11}$$

If, now, the substitutions

$$\left.\begin{aligned} \frac{\pi}{l} &= \lambda \\ l\delta &= \eta \end{aligned}\right\} \tag{12}$$

are made, equations (5) and (7) yield

$$\phi = \frac{U}{\sqrt{1-M_\infty^2}} \int_0^\infty \eta(\lambda) \frac{K_o\left(\sqrt{1-M_\infty^2}\lambda r\right)}{K_1\left(\sqrt{1-M_\infty^2}\lambda a\right)} \sin(\lambda x) d\lambda \tag{13}$$

The corresponding solution for incompressible flow is

$$\phi_i = U\int_0^\infty \eta(\lambda) \frac{K_o(\lambda r)}{K_1(\lambda a)} \sin(\lambda x) d\lambda \tag{14}$$

A comparison of equations (14), (13), and (11) shows that

$$\phi_c(x,r) = \frac{1}{1-M_\infty^2} \phi_i \left(x, \sqrt{1-M_\infty^2} r\right) \tag{15}$$

which is in accordance with the general results for this case (reference 2). The velocity correction formula which follows from equation (13) is of the form

$$\frac{u_c(x,a)}{u_i(x,a)} = \frac{1}{\sqrt{1-M_\infty^2}} \frac{\displaystyle\int_0^\infty \lambda\eta(\lambda) \frac{K_o\left(\sqrt{1-M_\infty^2}\lambda a\right)}{K_1\left(\sqrt{1-M_\infty^2}\lambda a\right)} \cos(\lambda x) d\lambda}{\displaystyle\int_0^\infty \lambda\eta(\lambda) \frac{K_o(\lambda a)}{K_1(\lambda a)} \cos(\lambda x) d\lambda} \tag{16}$$

VELOCITY CORRECTION FORMULA FOR RIPPLE IN THE PRESENCE OF TUNNEL WALLS

If, again, a ripple of the form of equation (4) is taken with the boundary condition (equation (3)) at the surface of the body of revolution, there is now an additional condition at the boundary of a tunnel of radius b, namely, for r = b

$$\frac{\partial\phi}{\partial r} = 0 \tag{17}$$

The perturbation potential ϕ which satisfies equations (3) and (17) is of the following form:

$$\phi = \frac{Ul\delta}{\sqrt{1-M_\infty^2}} \frac{I_1(\beta b)K_0(\beta r) + K_1(\beta b)I_0(\beta r)}{I_1(\beta b)K_1(\beta a) - K_1(\beta b)I_1(\beta a)} \sin(\alpha x) \tag{18}$$

where

$$\left.\begin{aligned} \alpha &= \frac{\pi}{l} \\ \beta &= \sqrt{1-M_\infty^2}\,\frac{\pi}{l} \end{aligned}\right\} \tag{19}$$

When $b \longrightarrow \infty$, equation (18) reduces to equation (7). The axial perturbation velocity u at the surface of the body of revolution follows from equation (18) in the form

$$u(x,a) = \frac{\pi U\delta}{\sqrt{1-M_\infty^2}} \frac{I_1(\beta b)K_0(\beta a) + K_1(\beta b)I_0(\beta a)}{I_1(\beta b)K_1(\beta a) - K_1(\beta b)I_1(\beta a)} \cos(\alpha x) \tag{20}$$

On the basis of equation (20), the values of $\frac{u_c}{u_i}$ may, for given values of $\frac{a}{l}$, $\frac{b-a}{l}$, and M_∞, be calculated numerically.

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National Advisory Committee for Aeronautics
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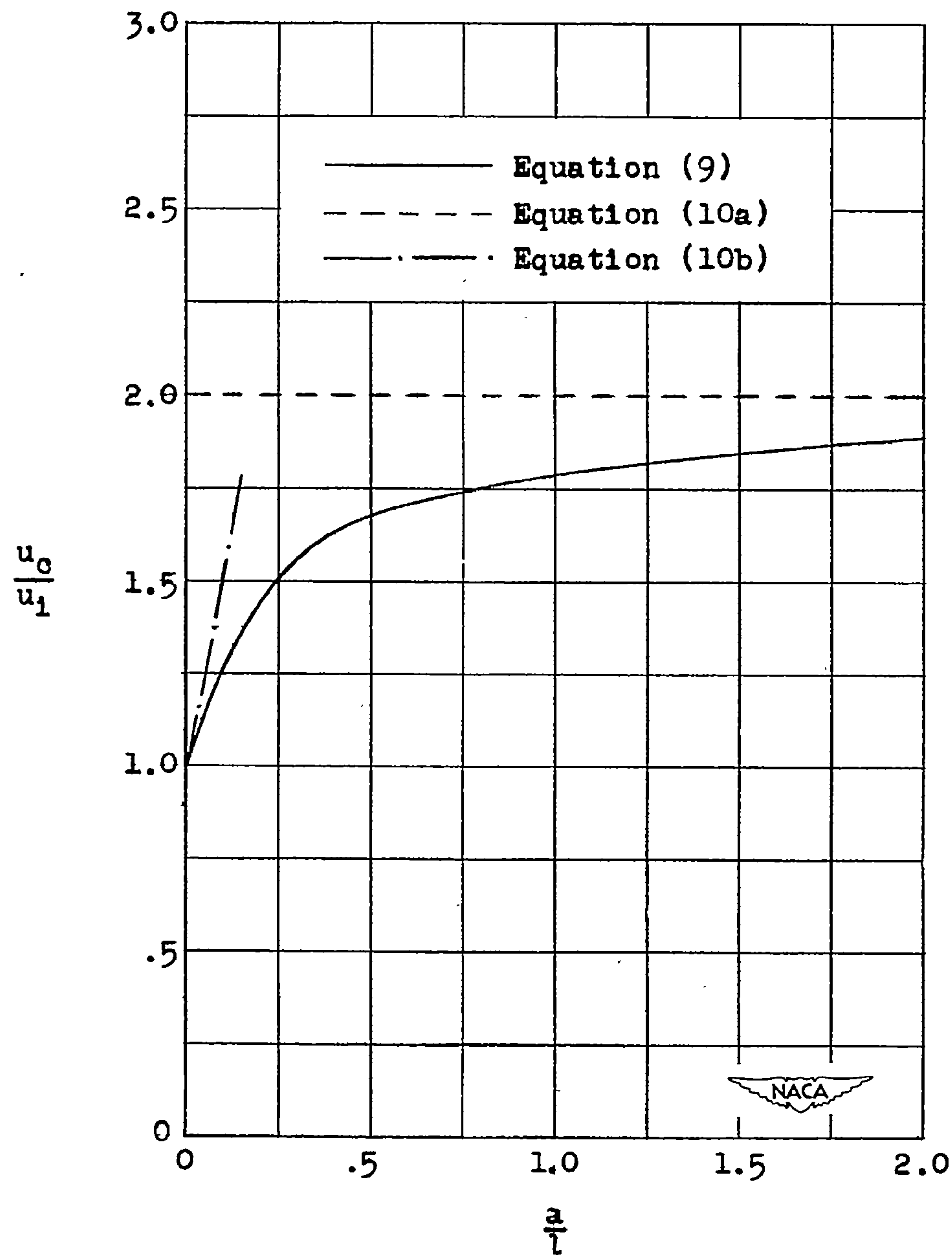

Figure 2.- Comparison of compressibility corrections. $M_{\infty} = 0.866$.